COMMUTATIVE ALGEBRA - MMATH SECOND YEAR - FINAL EXAMINATION

27th November, 2012 - 10 a.m. to 1 p.m. - Total Marks: 50

Attempt all questions. All rings considered here are commutative rings with multiplicative identity. You can quote any result proved in Atiyah-Macdonald but you cannot quote problems.

(1) Any ring is a $\mathbb{Z}$-algebra as there always exists a (unique) ring homomorphism $\mathbb{Z} \to R$. A ring R is said to be *finitely generated* if it is a finitely generated as a $\mathbb{Z}$-algebra. Prove that, if a finitely generated ring R is a field, then it is a finite field. (Hint: The homomorphism $\mathbb{Z} \to R$ is either injective or not, treat both cases).(**9 marks**)

(2) (a) Let k be a field, let $k[x, y]$ be the polynomial ring in two variables and let $A \subseteq k[x.y]$ be the subring $A := \{f(x, y) = a + xg(x, y) | a \in k, g(x, y) \in k[x, y]\}$. That is, A consists of all polynomials in which the coefficient of every pure power y^i of y (for all $i > 0$) is zero. Prove that A is *not* a Noetherian ring.

(b) Consider the ring extension $\mathbb{Z} \subseteq \mathbb{Z}[1/3]$. Is this an integral extension?

(c) Prove that $k[x^2] \subseteq k[x]$ is an integral extension of rings (here k is a field and x an indeterminate). For any $f(x) \in k[x]$, write down an explicit equation of integral dependance of f over $k[x^2]$.

(**4+4+4=12 marks**)

(3) Let R be a Noetherian ring, M a nonzero finitely generated R-module and P a prime ideal of R containing the ideal $Ann(M)$. Prove that, P is minimal among prime ideals containing $Ann(M)$, if and only if, the R_P module M_P is a nonzero module of finite length. (**10 marks**)

(4) Let A be a ring, x denote an indeterminate, $A[x]$ denote the ring of polynomials in x with coefficients from A. For any ideal I of A, let $I[x]$ denote the ideal of $A[x]$ consisting of polynomials in x with coefficients from I.

(a) If P is a prime ideal of A, prove that $P[x]$ is a prime ideal of $A[x]$.

(b) If $Q \subset A$ is P-primary, show that $Q[x] \subset A[x]$ is $P[x]$-primary.

(c) If $I = \cap_{i=1}^{n} Q_i$ is a minimal primary decomposition of I in A (where Q_i is P_i-primary for every i), then prove that $I[x] = \cap_{i=1}^{n} Q_i[x]$ is a minimal primary decomposition of $I[x]$ in $A[x]$.

(**2+4+4 = 10 marks**)

(5) A ring homomorphism $f : A \to B$ is said to have the *going-up property* if the conclusion of the going-up theorem holds for B and its subring $f(A)$. Let $f^* : Spec(B) \to Spec(A)$ be the mapping associated to f. Consider the following three statements

(a) f^* is a closed mapping.

(b) f has the going-up property.

(c) Let Q be any prime ideal of B, and let $P = Q^c$. Then $f^* : Spec(B/Q) \to Spec(A/P)$ is surjective.

. Prove that (a) implies (b), and (b) is equivalent to (c). (**3+3+3=9 marks**)